Assignment 9: Sequences, Series, Taylor Polynomials, and Taylor Series
Week 13

In class on Wednesday, [Day 35]:

- Lecture on themes of Chapter 10: Sequences, series, and Taylor polynomials
- Homework: [Note: Homework #1 and Homework #2 follow on the next couple of pages.]
  Several of you asked for some homework problems that I would collect.
  - Do Homework #1, which is available as a separate handout in the Assignments folder. Look for a file named HWK1_TaylorPolynomials. This homework set is due at the beginning of class on Friday.
  - Continue to work on the Activities and Checkpoints of Sections 10.1 – 10.4. The answers to these problems are given in the text. Be sure to ask me about any problems that you find confusing.
  - Continue to work on Web Work problems, which are DUE at 11:30 pm on Friday.

In class on Friday, [Day 36]:

- Section 10.5: *Series of Constants*
- Lecture / large class discussion of Section 10.5
- Homework:
  - Do Homework #2, which is available as a separate handout in the Assignments folder. Look for a file named HWK2_TaylorPolynomials. This homework set is due at the beginning of class on Monday.
  - Study Section 10.5, and attempt the Activities and Checkpoints on your own. The answers to these problems are given in the text. Come to class on Monday prepared to ask me about those problems that you find confusing.
  - Project 4 will be available. (Look in the Projects folder under COURSE MATERIALS for this course in Educator.) This project takes another look at the Drug Dosage problem using geometric series to find a solution.

Web Work problem sets:

- Web Work problems sets Ch10_Limits (A – E) are available, and due **[Day 36]** (at 11:30 pm). Working on these problems will help you begin to prepare for Benchmark #3, which will be given in class on Monday, **[Day 40]**.
- To get credit for these problem sets, you need to complete 80% of these problems correctly.

In class on Monday, [Day 37]:

- Section 10.6: *Convergence of Series*
- Lecture / large class discussion of Section 10.6.
- Homework:
  - Study Section 10.6, and attempt the Activities and Checkpoints on your own. The answers to these problems are given in the text. Come to class on Wednesday prepared to ask me about those problems in Sections 10.5 and 10.6 that you find confusing.
  - The Study Guide for Benchmark #3 will be available on Monday afternoon. This Benchmark will be given in class on Monday, **[Day 40]**. The window-of-opportunity for passing this Benchmark is **[Day 40 – 43]**.

Note: This page contains the text for two short handouts with problems about constructing Taylor polynomials.

Mt 211 Calculus II
Homework #1: Constructing Taylor Polynomials
DUE: [Day 36]

Directions: *Write out your solution to this problem. Show your work so that I can follow your thinking process. You may discuss this problem with your classmates, but each of you is to turn in your work individually. This problem is due at the beginning of class on [Day 36].*

Consider the function $f(x) = 5x^3 - 3x^2 + 4x - 7$.

1. Compute the fourth-degree Taylor polynomial for f, which is centered at $x = 0$. Show your work by setting up a table (as I've done on the board several times).

2. What can you say in general about the m^{th} degree Taylor polynomial at $x = 0$ for a polynomial of degree n?

Mt 211 Calculus II
Homework #2: Constructing Taylor Polynomials
DUE: [Day 37]

Directions: *Write out your solution to this problem. Show your work so that I can follow your thinking process. You may discuss this problem with your classmates, but each of you is to turn in your work individually. This problem is due at the beginning of class on [Day 37].*

The tangent line to the graph of a function f at $x = a$ is the first-degree Taylor polynomial for f at $x = a$. Similarly, a certain close-fitting parabola is the second-degree Taylor polynomial of f at $x = a$.

3. Consider the function $f(x) = \sqrt{25 - x^2}$. Describe the graph of this function. (This is one of the functions on the list of functions for the Study Guides for our Tests. You should be able to describe this graph simply in words.

4. Compute the second degree Taylor polynomial for f, which is centered at $x = 0$. Show your work by setting up a table (as I've done on the board several times).

5. If a Taylor polynomial is constructed around a different center, say $x = a$, the form of the polynomial is slightly different:
   $$P(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \text{ where } c_n = \frac{f^{(n)}(a)}{n!}.$$
   Construct the second-degree Taylor polynomial for f which is centered at $x = 4$ instead of $x = 0$.

6. Use Maple to get graphs of f, and the two Taylor polynomials on the same pair of axes. The relevant syntax for plotting these functions is as follows:
   $f := x \rightarrow \mathrm{sqrt}(25 - x^2);$
   $P1 := x \rightarrow (\mathit{your\ polynomial});$
   $P2 := x \rightarrow (\mathit{your\ polynomial});$
   $\mathit{plot}([f(x), P1(x), P2(x)], x = -a..a);$

Teaching Notes for [Day 35]: Sequences, Series, Taylor Polynomials, and Taylor Series

In class on [Day 35]:

- **Some limit problems on the board** (Use problems from Ch1_Limits_E.)
  - Limits have us look in the neighborhood around a point – not directly at the point.
  - Strategies for limits as $x \to a$:
    - Try plugging in values – but beware of places where there is a 0 in the denominator!
    - $A/_0 \to \infty$, but for $0/_0$ anything can happen!
    - If factoring eliminates the point where the function is undefined, the simplified function is the same as the original function EXCEPT at the point that was eliminated. There is a "hole" in the original function.
    - If we cannot eliminate the problem by factoring, look for a vertical asymptote. Need to check the behavior on either side of the problem point.
    - For functions defined over a split domain (or functions defined in parts), need to consider the behavior on each side of the seam points. Limiting behavior is about the behavior as we approach the seam point – NOT about the behavior as the seam point.
  - Strategies for limits as $x \to \infty$:
    - Consider behavior for very large values of x or $-x$.
    - Look at the highest-powered term in numerator over highest-powered term in denominator.

- **What we Taylor polynomials – why do we care?**
  - In Calculus I we found the tangent line to a curve at a point, $x = a$:
    - Use point-slope form of equation for a line: $(a, f(a))$ is the point; $f'(a)$ is the slope
  - The Taylor polynomial of degree n is a polynomial of degree n centered at point $x = a$ which hugs the curve at the point where $x = a$. This is an extension of the idea of finding the tangent line to a curve.
    - <u>Example</u>: Develop a Taylor polynomial for $f(x) = \ln(1 + x)$ at the point $x = 0$.
      $P(x) = 0 + 1\,x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ \ldots$
    - In Activity 1, Section 10.4.1, we found that the Taylor polynomial for $g(x) = \frac{1}{1-x}$ is $P_1(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$.
      Now consider developing the Taylor polynomial for $g_2(x) = \frac{1}{1+x} = \frac{1}{1-(-x)}$. (See Section 10.4.2)
      We can work from scratch or we can substitute $x = -x$ in the polynomial we just developed:
      $P_2(-x) = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots = 1 - x + x^2 - x^3 + x^4 - \ \ldots$.
    - Suppose we find the integral $\int_0^x \frac{1}{1+t}\,dt = \ln(1 + x) - \ln(1 + 0) = \ln(1 + x)$.
      Sure enough, $\int_0^x P(x) = P_2(x)$.

    - ➢ **So we don't always have to develop the Taylor polynomial from scratch!**

- <u>Homework</u>:
  Several of you asked for some homework problems that I would collect.
  - Homework #1 is available in the Assignments folder: HWK1_TaylorPolynomials. DUE at the beginning of class on [Day 36].
  - There are some Web Work problems on limits, which are DUE at 11:30 pm on Friday.
  - Homework #2 is available in the Assignments folder: HWK2_TaylorPolynomials. DUE at the beginning of class on [Day 37].
  - Continue to work on the Activities and Checkpoints of Sections 10.1 – 10.4. The answers to these problems are given in the text. Be sure to ask me about any problems that you find confusing.

In class on [Day 36]:

- I will introduce Section 10.5: *Series of Constants* with lecture / discussion.

Teaching Notes for [Day 36]: Sequences and Series of Numbers

- Collect HWK-1.
  - Can someone explain briefly what happens in this problem?
  - What is the Taylor polynomial of degree m for $f(x) = 5x^3 - 3x^2 + 4x - 7$? Why does this happen?
- Homework:
  Several of you asked for some homework problems that I would collect.
  - There are some Web Work problems on limits, which are DUE at 11:30 pm tonight.
  - Homework #2 is available in the Assignments folder: HWK2_TaylorPolynomials. DUE at the beginning of class on Monday.
  - Project 4: Drug Dosage (reprise) – There is a geometric series in the concentrations of the drug in the body. If you can identify this series, you can refine your earlier solution. DUE May 5.
  - In class today, we will be doing Section 10.5, so continue to work on the Activities and Checkpoints of the text up through Section 10.5. The answers to these problems are given in the text. Be sure to ask me about any problems that you find confusing.
  - On Monday: Section 10.6, but you will have time to ask questions about issues in Section 10.5.

- Introduce Section 10.5: *Series of Constants* with lecture / discussion:
  - Language: sequence (infinite list), series (infinite sum), terms of the sequence, partial sums of the series
  - Examples:
    - Geometric series: $a + ar + ar^2 + ar^3 + \ldots$
      - Sum of geometric series: $S = a/(1-r)$, where a is the first term and r is the common ratio (See Section 10.1)
    - Taylor polynomial (see notes from last class)
      $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}$, converges for $|x| < 1$
    - What happens if we evaluate this polynomial at $x = 2$? (Activity 1, Section 10.5.1)

$$ln(1+2) = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{3} + \frac{2^5}{5} - \frac{2^6}{6} + \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{2^{k+1}}{k+1} = 2 - 2 + \frac{8}{3} - \frac{16}{4} + \frac{32}{5} - \ldots$$

      - Does this converge to a finite number? … diverge to infinity? … oscillate between two finite numbers? … oscillate between larger and larger values (possibly of alternate signs)?
      - To study this behavior, consider the real-valued function $f(x) = \frac{2^x}{x}$. What happens to this function as $x \to \infty$? Why?
      - Now consider the discrete function $f(n) = (-1)^n \frac{2^{n+1}}{n+1}$.
        Alternating series
        Individual terms are going to $\pm\infty$, so the series cannot converge to a finite number. In fact, it oscillates between larger and larger numbers.

  - Divergence Test: If the sequence $b_0, b_1, b_2, b_3, b_4, \ldots$ does not converge to 0, then the sequence of partial sums $b_0, b_0 + b_1, b_0 + b_1 + b_2, b_0 + b_1 + b_2 + b_3, b_0 + b_1 + b_2 + b_3 + b_4, \ldots$
    series (infinite sum) $\sum_{k=0}^{\infty} b_k$ diverges.

  - More language: alternating series, alternating harmonic series, harmonic series
  - The alternating harmonic series arose from the Taylor polynomial for $ln(1+x)$, and the authors claim that this polynomial converges when $|x| < 1$.
    - What happens to the harmonic series? (Activity 2, Section 10.5.2)
    - What is really going on for the alternating harmonic series? (Activity 3, Section 10.5.3 – two different experiments to help us see what is going on)
  - Taylor polynomial for $arctan(x) = x - x^3/3 + x^5/5 - \ldots$ (See Section 10.4.3)
  - The Leibnitz Series: $arctan(1) = 1 - 1^3/3 + 1^5/5 - \ldots$ = 1 – 1/3 + 1/5 - 1/7 + 1/9 - …
    The $arctan\ (1)$ means the "angle whose tangent is 1," so this ought to equal $\pi/4$.
    Does the Leibnitz Series really add up to $\pi/4$? (Activity 4, Section 10.5.4)

Teaching Notes for [Day 37]: Everything you want to know about Sequences and Series

- Collect HWK-2.
  - Can someone briefly summarize what happens in this problem?
  - What are the two Taylor polynomials you developed? Were you "surprised" by the shapes of the two parabolas? Where is the vertex of the parabola in each case?

- Comment on HWK-1.
  - In question 1, you were asked to find the Taylor polynomial of degree 4 for a particular polynomial. In questions 2, you were asked to say something "in general" about the Taylor polynomial of degree m for a polynomial of degree n. Note that we have to think about three cases: (1) $m > n$, (2) $m = n$, and (3) $m < n$. What am I asking you to observe (or "see") in this situation?

- Homework:
  - Due date for Web Work problems on limits was extended to 11:30 pm tonight.
  - Study Guide for Benchmark #3 is available. This Benchmark will be given at the end of class next Monday.
  - New Web Work problem sets (B3 A – F) to help you practice for this Benchmark.
  - Study the Activities and Checkpoints of Sections 10.5, and 10.6 – How do we know when a series converges or diverges?

- Project 4: Drug Dosage (reprise) – There is a geometric series in the concentrations of the drug in the body. If you can identify this series, you can refine your earlier solution. DUE [Day 41]
  - Questions about the project? … Our work in Section 10.6 will include some examples where we are finding a common ratio for the terms in a series.

- Summarize important ideas from Section 10.5: *Series of Constants*
  - Language: sequence (infinite list), series (infinite sum), terms of the sequence, partial sums of the series
  - Geometric series: $a + ar + ar^2 + ar^3 + \ldots$
    - Sum of geometric series: $S = a/(1-r)$, where a is the first term and r is the common ratio (See Section 10.1)
  - Divergence Test: If the sequence $b_0, b_1, b_2, b_3, b_4, \ldots$ does not converge to 0, then the series (infinite sum) $\sum_{k=0}^{\infty} b_k$ diverges.

  - The alternating harmonic series arose from the Taylor polynomial for $ln(1+x)$, and the authors claim that this polynomial converges when $|x| < 1$.
    - What happens to the harmonic series? (**Activity 2, Section 10.5.2**)
    - Taylor polynomial for $ln(1+x)$ converges when $|x| < 1$, and diverges elsewhere
    $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$
    - What is really going on for the alternating harmonic series? For what values of x does this series converge? (**Activity 3, Section 10.5.3** – two different experiments to help us see what is going on)
    - Taylor polynomial for $\arctan(x) = x - x^3/3 + x^5/5 - \ldots$ (See Section 10.4.3) For what values of x does the $\arctan(x)$ converge?
    - The Leibnitz Series: $\arctan(1) = 1 - 1^3/3 + 1^5/5 - \ldots$ = 1 – 1/3 + 1/5 - 1/7 + 1/9 - … The $\arctan(1)$ means the "angle whose tangent is 1," so this ought to equal $\pi/4$. Does the Leibnitz Series really add up to $\pi/4$? (**Activity 4, Section 10.5.4**)

- Section 10.6: *Convergence of Series*
  Take some time to do these activities in small groups, and summarize the discussion for the whole class.
  - Alternating Series Test (Section 10.6.1): Read this together – and make connections to the *harmonic series* and the *alternating harmonic series*.

- Alternating Series Test (**Activity 1, Section 10.6.1**)
  - The "tail" of a sequence or series
  - Use the Alternating Series Test to find the interval of convergence for the $\arctan(x)$ function (**Section 10.6.2 – Example 1, Checkpoint 1**)
  - Use the Alternating Series Test to find the interval of convergence for the logarithmic function, $\ln(1 + x)$ (**Section 10.6.3 – Activity 2**)

- Complete Section 10.6 on Wednesday
  - Using the geometric series to estimate tails
  - Power Series
  - How fast does $n!$ grow?
  - How do we know that the Taylor Series for sin(x) and cos(x) converge? What is their interval of convergence?
  - Ratio test